

Asymptotic expansions for enumerating connected labelled graphs

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Abstract

I compute several terms of the asymptotic expansion of the number of connected labelled graphs with n nodes and m edges, for small $k = m - n$. I thus identify an error in a recent paper of Flajolet *et al.*

1 Introduction

Consider the problem of computing the number $c(n, m)$ of connected labelled graphs with n nodes and $m = n - 1, n, n + 1, \dots$ edges, for fixed small m as $n \rightarrow \infty$ ([slo03], sequence A057500, and links therein). From this, we can compute the probability of a randomly chosen labelled graph being connected, which is useful in various applications to communication networks.

In a recent paper Flajolet *et al.* [fss04], the authors used ingenious analytic methods to compute that the number of labelled connected graphs with n nodes and excess $= k \geq 2$ (excess is defined as the number of edges minus the number of nodes) is asymptotically

$$A_k(1)\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)}+\frac{A'_k(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{\frac{2}{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

where $A_k(1)$ is given in terms of Airy functions; the first few values being as in table 1.

However, I found that this result agreed poorly in comparisons with exact counts, which can be easily computed from the known generating functions. I

| k | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|-------|-------|-----------|-----------|-------------|---------------|-------------------|
| $A_k(1)$ | 5/24 | 5/16 | 1105/1152 | 565/128 | 82825/3072 | 19675/96 | 1282031525/688128 |
| $A'_k(1)$ | 19/24 | 65/48 | 1945/384 | 21295/768 | 603965/3072 | 10454075/6144 | 1705122725/98304 |

Table 1: Coefficients in the formula of Flajolet *et al.*

| k | type | $[n^0]$ | $[n^{-1/2}]$ | $[n^{-1}]$ | $[n^{-3/2}]$ | $[n^{-2}]$ | $[n^{-5/2}]$ |
|-----|-------------|---------------------|------------------------|-------------------------|----------------------|----------------------|--------------------|
| 0 | unicycle | $\xi \frac{1}{4}$ | $-\frac{7}{6}$ | $\xi \frac{1}{48}$ | $\frac{131}{270}$ | $\xi \frac{1}{1152}$ | $-\frac{4}{2835}?$ |
| 1 | bicycle | $\frac{5}{24}$ | $-\xi \frac{7}{24}$ | $\frac{25}{36}$ | $-\xi \frac{7}{288}$ | $-\frac{79}{3240}?$ | |
| 2 | tricycle | $\xi \frac{5}{256}$ | $-\frac{35}{144}$ | $\xi \frac{1559}{9216}$ | $-\frac{55}{144}$ | | |
| 3 | quadricycle | $\frac{221}{24192}$ | $-\xi \frac{35}{1536}$ | | | | |

Table 2: Coefficients in the series for $c(n, n+k)/n^{n+(3k-1)/2}$, conjectured from numerical experiments. Here and elsewhere $[n^x]f(n)$ means the coefficient of n^x in $f(n)$.

did comparisons with exact counts for up to $n = 1000$ nodes and for excess $k = 2, 3, \dots, 8$. The results led me to suspect that the factor $(n/e)^n$ should be simply n^n , and that the second term in the square brackets should have a minus sign. The purpose of this note is therefore to compute more terms of the asymptotic expansions (by different methods) to confirm these suspicions. The first term in these asymptotic expansions was already known thanks to a recurrence relation (involving an implicitly defined quantity) due to Bender *et al.* [bcm90].

The asymptotic expansion of the number $c(n, n+k)$ of labelled connected graphs with n nodes and excess $k \geq -1$ has the form

$$c(n, n+k) \sim n^{n+(3k-1)/2} \sum_{j=0}^{\infty} a_j(k) \xi^{1-((k+j) \bmod 2)} n^{-j/2}$$

where $\xi \equiv \sqrt{2\pi}$, and the coefficients $a_j(k)$ are rational. This structure in fact allows surprisingly reliable estimates of $a_j(k)$ for $j+k$ less than about 5 simply by fitting least-squares polynomials in $x = n^{-1/2}$ to the exact data. By this means I obtained the estimates shown in table 2.

| k | type | B: $[n^0]$ | F: $[n^0]$ | F: $[n^{-1/2}]$ | $[n^{-1}]$ |
|-----|-------------|--------------------------|--------------------------|-----------------------|------------|
| 0 | unicycle | $\xi \frac{1}{4}$ | | | |
| 1 | bicycle | $\frac{5}{24}$ | | | |
| 2 | tricycle | $\xi \frac{5}{256}$ | $\xi \frac{5}{256}$ | $\frac{35}{144}$ | |
| 3 | quadracycle | $\frac{221}{24192}$ | $\frac{221}{24192}$ | $\xi \frac{35}{1536}$ | |
| 4 | pentacycle | $\xi \frac{113}{196608}$ | $\xi \frac{113}{196608}$ | $\frac{221}{20736}$ | |

Table 3: Some (possibly incorrect) coefficients for $c(n, n+k)/n^{n+(3k-1)/2}$ from the literature. B: from [bcm90]; F: from [fss04] (with removal of factor e). Here $\xi \equiv \sqrt{2\pi}$. Note that $c(n, n-1)/n^{n-2} = 1$. Missing values are not available in the literature.

2 Generating functions

The exponential generating function (egf) for labelled graphs is

$$g(w, z) = \sum_{n=0}^{\infty} (1+w)^{\binom{n}{2}} z^n / n!.$$

This means that $n! [w^m z^n] g(w, z)$ is the number of labelled graphs with m edges and n nodes. The exponential generating function for all connected labelled graphs is therefore

$$\begin{aligned} c(w, z) &= \log(g(w, z)) \\ &= z + w \frac{z^2}{2} + (3w^2 + w^3) \frac{z^3}{6} + (16w^3 + 15w^4 + 6w^5 + w^6) \frac{z^4}{4!} + \dots \end{aligned}$$

I will now compute the asymptotic expansion of $c(n, n+k)/n^{n+\frac{3k-1}{2}}$. This needs some preliminary manipulations concerning the quantities Q , W and t .

2.1 Q

The results in this section follow from theory available in [jklp93] and [fgkp95]. Ramanujan's Q -function [ram11] is defined for $n = 1, 2, 3, \dots$ by

$$Q(n) \equiv \sum_{k=1}^{\infty} \frac{n^k}{n^k} = 1 + \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \dots,$$

We have $\sum_{n=1}^{\infty} Q(n) n^{n-1} \frac{z^n}{n!} = -\log(1-T(z))$, where $T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n = z \exp(T(z))$ is the egf for rooted labelled trees. To get the large- n asymptotics of

| | $[n^{1/2}]$ | $[n^0]$ | $[n^{-1/2}]$ | $[n^{-1}]$ | $[n^{-3/2}]$ | $[n^{-2}]$ |
|-----|-----------------|----------------|------------------|--------------------|--------------------|--------------------------|
| D | 0 | $\frac{2}{3}$ | $\frac{8}{135}$ | $-\frac{16}{2835}$ | $-\frac{32}{8505}$ | $\frac{17984}{12629925}$ |
| Q | $\frac{\xi}{2}$ | $-\frac{1}{3}$ | $\frac{\xi}{24}$ | $-\frac{4}{35}$ | $\frac{\xi}{576}$ | $\frac{8}{235}$ |

Table 4: Coefficients in asymptotic expansions of D and Q .

Q , first consider the related function [fgkp95]

$$R(n) \equiv 1 + \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \dots, \quad n = 1, 2, 3, \dots$$

and let $D(n) = R(n) - Q(n)$. We may immediately deduce:

1. $Q(n) + R(n) = n! e^n / n^n$
2. $\sum_{n=1}^{\infty} D(n) n^{n-1} \frac{z^n}{n!} = \log\left[\frac{(1-T(z))^2}{2(1-ez)}\right]$
3. $D(n) \sim \sum_{k=1}^{\infty} c(k) [z^n] (T(z)-1)^k$, where $c(k) \equiv [\delta^k] \log(\delta^2 / (2(1-(1+\delta)e^{-\delta})))$
4. $D(n) \sim \frac{2}{3} + \frac{8}{135} n^{-1} - \frac{16}{2835} n^{-2} - \frac{32}{8505} n^{-3} + \frac{17984}{12629925} n^{-4} + \frac{668288}{492567075} n^{-5} + O(n^{-6})$

Now using $Q(n) = (n! e^n / n^n - D(n)) / 2$, we get the results shown in table 4.

2.2 W

Now let W_k be the egf for connected labelled $(k+1)$ -cyclic graphs. It is known that [jklp93]:

1. for unrooted trees $W_{-1}(z) = T(z) - T^2(z)/2$, $[z^n]W_{-1}(z) = n^{n-2}$
2. for unicycles $W_0(z) = -(\log(1-T(z)) + T(z) + T^2(z)/2)/2 = \frac{1}{3!}z^3 + \frac{15}{4!}z^4 + \frac{222}{5!}z^5 + \frac{3660}{6!}z^6 + \dots$
3. for bicycles $W_1(z) = \frac{6T^4(z) - T^5(z)}{24(1-T(z))^3} = \frac{6}{4!}z^4 + \frac{205}{5!}z^5 + \frac{5700}{6!}z^6 + \dots$
4. for $k \geq 1$, $W_k(z) = \frac{A_k(T(z))}{(1-T(z))^{3k}}$, where A_k are polynomials explicitly computable from results in [jklp93]

| k | $[n^0]$ | $[n^{-1/2}]$ | $[n^{-1}]$ | $[n^{-3/2}]$ | $[n^{-2}]$ | $[n^{-5/2}]$ |
|-----|---------------------|---------------------|-------------------------|----------------------|----------------------------|-------------------------|
| 0 | $\xi \frac{1}{4}$ | $-\frac{7}{6}$ | $\xi \frac{1}{48}$ | $\frac{131}{270}$ | $\xi \frac{1}{1152}$ | $-\frac{4}{2835}$ |
| 1 | $\frac{5}{24}$ | $-\xi \frac{7}{24}$ | $\frac{25}{36}$ | $-\xi \frac{7}{288}$ | $-\frac{79}{3240}$ | $-\xi \frac{7}{6912}$ |
| 2 | $\xi \frac{5}{256}$ | $-\frac{35}{144}$ | $\xi \frac{1559}{9216}$ | $-\frac{55}{144}$ | $\xi \frac{33055}{221184}$ | $-\frac{41971}{136080}$ |

Table 5: The asymptotic number of connected graphs: coefficients in the asymptotic expansion of $c(n, n+k)/n^{n+(3k-1)/2}$.

2.3 t

Knuth and Pittel's tree polynomials $t_n(y)$ ($y \neq 0$) are defined by

$$(1-T(z))^{-y} = \sum_{n=0}^{\infty} t_n(y) \frac{z^n}{n!}.$$

We can compute these for $y > 0$ from the recurrence

$$\begin{aligned} t_n(1) &= 1 \\ t_n(2) &= n^n(1+Q(n)) \\ t_n(y+2) &= (n/y) t_n(y) + t_n(y+1), \quad y > 0 \end{aligned}$$

Thanks to this recurrence, the asymptotics for each t_n follows from the known asymptotics of Q . To apply these results to the problem of asymptotically expanding $c(n, n+k)$, we need to express $c(n, n+k)$ as a linear combination of values of $t_n(l)$ at integers l , which is always possible by solving a linear system. Some examples for small k follow.

$$\begin{aligned} c(n, n) &= n![z^n]W_0(z) \\ &= \frac{1}{2}Q(n)n^{n-1} + 3/2 + t_n(-1) - t_n(-2)/4 \\ c(n, n+1) &= n![z^n]W_1(z) \\ &= \frac{5}{24}t_n(3) - \frac{19}{24}t_n(2) + \frac{13}{12}t_n(1) - \frac{7}{12}t_n(0) + \frac{1}{24}t_n(-1) + \frac{1}{24}t_n(-2) \end{aligned}$$

We finally have the desired exact results for $c(n, n+k)$ as shown in table 5. The conjectured numerical results of table 1 are confirmed and it appears that Flajolet *et al.* are in error.

3 Probability of connectivity

We now have all the results needed to calculate the asymptotic probability $P(n, n+k)$ that a randomly chosen graph with n nodes and $n+k$ edges is connected

(for $n \rightarrow \infty$ and small fixed k). The total number of graphs is $g(n, n+k) \equiv \binom{n}{n+k}$. This can be asymptotically expanded for small k . The results are in table 6. Thus we get the final results for the probability of connectivity in table 7.

| k | $[n^0]$ | $[n^{-1}]$ | $[n^{-2}]$ | $[n^{-3}]$ | $[n^{-4}]$ | $[n^{-5}]$ |
|-----|---------------------|------------------|-------------------|------------------------|------------------------|------------------------------|
| -1 | 1 | $\frac{7}{4}$ | $\frac{259}{96}$ | $\frac{22393}{5760}$ | $\frac{54359}{10240}$ | $\frac{52279961}{7741440}$ |
| 0 | $\frac{1}{2}$ | $-\frac{5}{8}$ | $-\frac{53}{192}$ | $-\frac{4067}{11520}$ | $-\frac{9817}{20480}$ | $-\frac{10813867}{15482880}$ |
| 1 | $\frac{1}{4}$ | $-\frac{21}{16}$ | $\frac{811}{384}$ | $-\frac{43187}{23040}$ | $\frac{159571}{73728}$ | $-\frac{55568731}{30965760}$ |
| k | $\frac{1}{2^{k+1}}$ | $\mathcal{O}(1)$ | | | | |

Table 6: The asymptotic total number of graphs: coefficients in the asymptotic expansion of $g(n, n+k)/(\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{(2k-1)/2})$.

| k | $[n^0]$ | $[n^{-1/2}]$ | $[n^{-1}]$ | $[n^{-3/2}]$ | $[n^{-2}]$ |
|-----|-----------------|--------------------|-------------------|----------------------|------------------------|
| -1 | $\frac{1}{2}$ | 0 | $-\frac{7}{8}$ | 0 | $\frac{35}{192}$ |
| 0 | $\frac{\xi}{4}$ | $-\frac{7}{6}$ | $-\frac{\xi}{3}$ | $-\frac{1051}{1080}$ | $\frac{5\xi}{9}$ |
| 1 | $\frac{5}{12}$ | $-\frac{7\xi}{12}$ | $\frac{515}{144}$ | $-\frac{28\xi}{9}$ | $\frac{788347}{51840}$ |

Table 7: The probability of connectivity: coefficients in the asymptotic expansion of $P(n, n+k)/(2^n e^{2-n} n^{k/2} \xi)$.

This method can easily be taken further further - it is simply a matter of symbol crunching. In figure 1, I confirm the accuracy of the new formulas by comparison with exact enumeration.

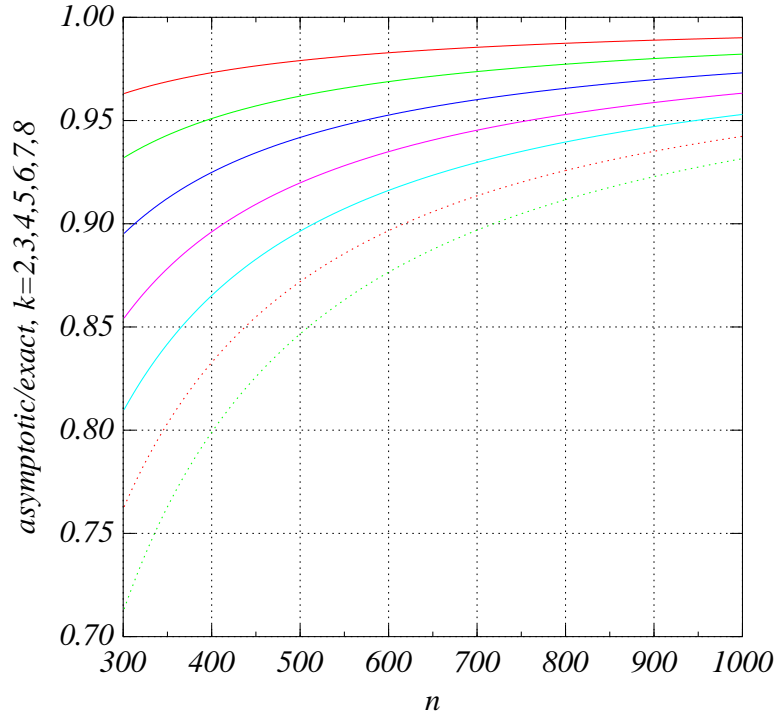


Figure 1: Enumeration of labelled graphs - comparison of exact data with new asymptotic formulas.

4 References

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